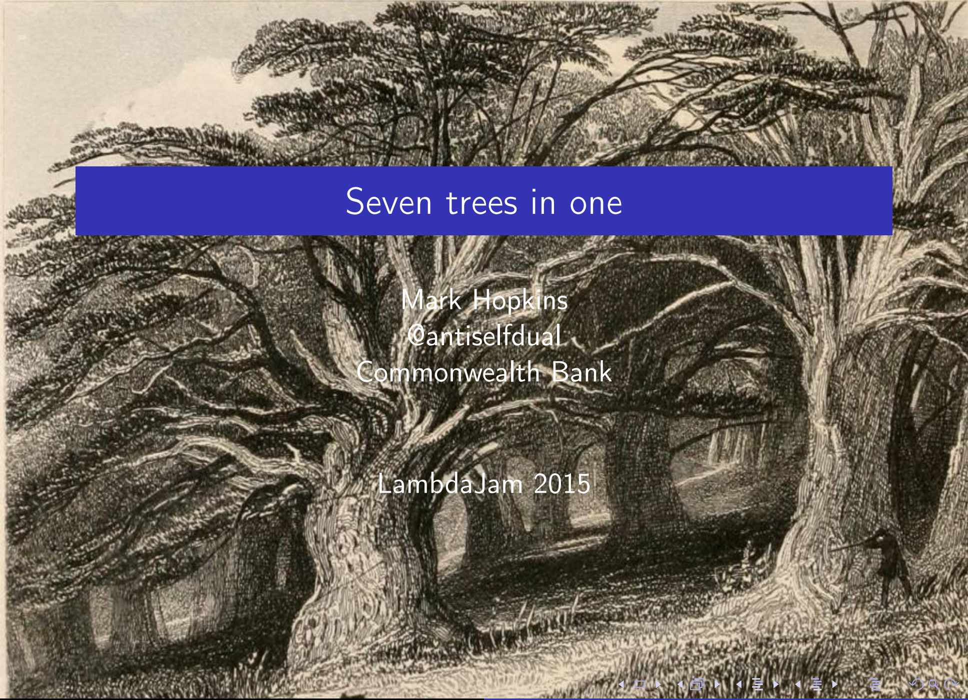

Seven trees in one

Mark Hopkins
@antiselfdual
Commonwealth Bank

LambdaJam 2015

Unlabelled binary trees

```
data Tree = Leaf | Node Tree Tree
```

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Suspend disbelief, and **solve for** T.

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$$\begin{aligned} T^2 - T + 1 &= 0 \\ T &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2} i \end{aligned}$$

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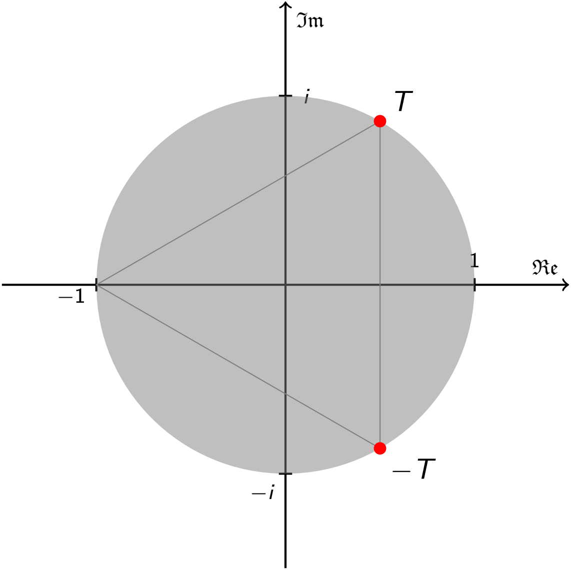
$$\begin{aligned} T^2 - T + 1 &= 0 \\ T &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2} i \\ &= e^{\pm \pi i/3} \end{aligned}$$



ℑm
i
T
1
ℜe
−1
−T
−i

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$$\Rightarrow \quad \text{true!}$$

Theorem

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Actually holds for any $k = 1 \mod 6$.
Not true for other values.

$T^2 \to T$

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Not surjective, since we can never reach `Leaf`.

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Not surjective either.

$T^2 \to T$, but cleverer

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f :: (Tree, Tree) → Tree
f (t1, t2) = go (Node t1 t2)
  where
        go t        = if leftOnly t then left t else t
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                   || right t == Leaf && leftOnly (left t)
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Bijective!

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```

Bijective! but not $O(1)$.

A solution

```
f :: T → (T, T, T, T, T, T, T)
f L                                                         = (L,L,L,L,L,L,L)
f (N t1 L)                                                  = (t1,N L L,L,L,L,L,L)
f (N t1 (N t2 L))                                           = (N t1 t2,L,L,L,L,L,L)
f (N t1 (N t2 (N t3 L)))                                    = (t1,N (N t2 t3) L,L,L,L,L,L)
f (N t1 (N t2 (N t3 (N t4 L))))                             = (t1,N t2 (N t3 t4),L,L,L,L,L)
f (N t1 (N t2 (N t3 (N t4 (N L L)))))                       = (t1,t2,N t3 t4,L,L,L,L)
f (N t1 (N t2 (N t3 (N t4 (N (N t5 L) L)))))                = (t1,t2,t3,N t4 t5,L,L,L)
f (N t1 (N t2 (N t3 (N t4 (N (N t5 (N t6 L)) L)))))         = (t1,t2,t3,t4,N t5 t6,L,L)
f (N t1 (N t2 (N t3 (N t4 (N (N t5 (N t6 (N t7 t8))) L))))) = (t1,t2,t3,t4,t5,t6,N t7 t8)
f (N t1 (N t2 (N t3 (N t4 (N t5 (N t6 t7))))))              = (t1,t2,t3,t4,t5,N t6 t7,L)
```

Where did this come from

$$T = 1 + T^2$$

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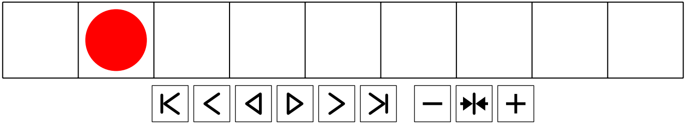
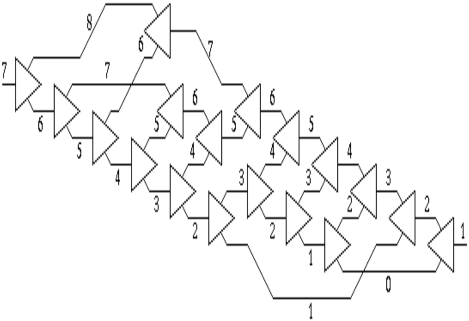
$$T^k = T^{k-1} + T^{k+1}$$

Penny game

T^0	T^1	T^2	T^3	T^4	T^5	T^6	T^7	T^8

- start with a penny in position 1.
- aim is to move it to position 7 by splitting and combining




8
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0
1

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And, under some conditions, the reverse implications hold.

Summary

- Simple arithmetic helps us find non-obvious type isomorphisms

Extensions

- Are there extensions to datatypes of decorated trees? (multivariate polynomials)
- What applications are there?

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- What applications are there?
  - important when writing a compiler to know when two types are isomorphic
  - It could interesting to split up a tree-shaped stream into seven parts

- Rich theory behind isomorphisms of polynomial types
- brings together a number of fields
  - distributive categories
  - theory of rigs (semirings)
  - combinatorial species
  - type theory

Further reading

- *Seven Trees in one*, Andreas Blass, Journal of Pure and Applied Algebra
- *On the generic solution to $P(X) = X$ in distributive categories*, Robbie Gates
- *Objects of Categories as Complex Numbers*, Marcelo Fiore and Tom Leinster
- *An Objective Representation of the Gaussian Integers*, Marcelo Fiore and Tom Leinster
- http://rfcwalters.blogspot.com.au/2010/06/robbie-gates-on-seven-trees-in-one.html
- http://blog.sigfpe.com/2007/09/arboreal-isomorphisms-from-nuclear.html

Challenge

Consider this datatype (Motzkin trees):

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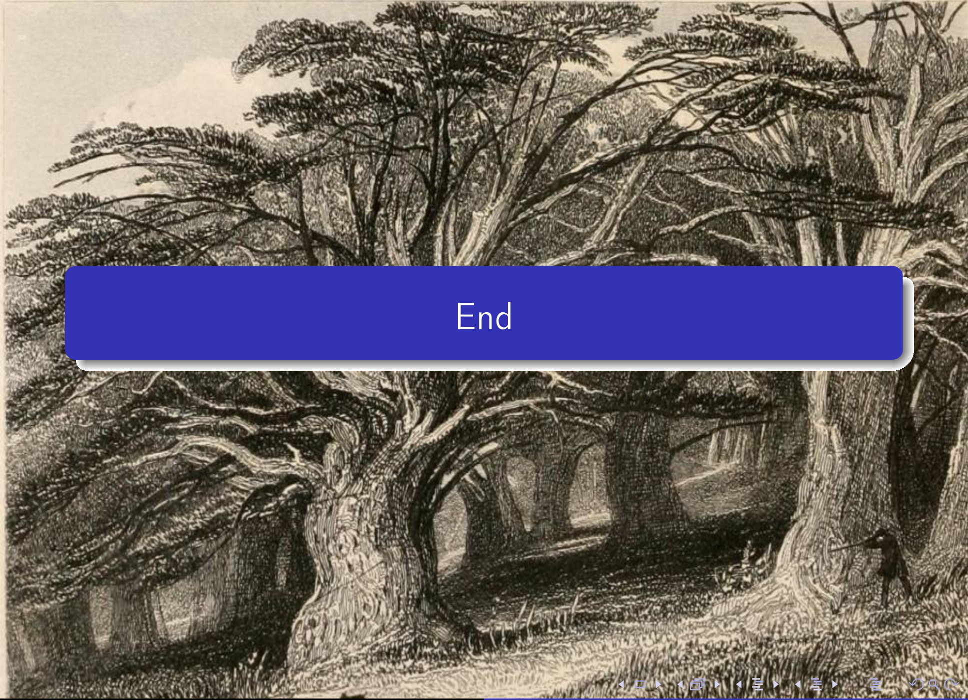
Show that $T^5 \cong T$

- by a nonsense argument using complex numbers
- by composing bijections (the penny game)
- implement the function and its inverse in a language of your choice

Photo credits

- *The Druid's Grove, Norbury Park: Ancient Yew Trees* by Thomas Allom 1804-1872 http://www.victorianweb.org/art/illustration/allom/1.html



End